J. Fluid Mech. (1997), *vol.* 342, *pp.* 403–409. Printed in the United Kingdom

An axisymmetric free surface with a 120 degree angle along a circle

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(Received 24 December 1996 and in revised form 3 March 1997)

An axisymmetric flow due to a submerged sink in water of infinite depth is considered, with a stagnation point on the free surface above the sink. Forbes & Hocking (1990) calculated numerically a flow for each value of the Froude number F smaller than a critical value F_c. For F close to F_c there is a ring-shaped bump on the free surface. At $F = F_c$, the crest of the bump becomes a ring of stagnation points. We use the numerical procedure of Hocking & Forbes to show that the bump is the first crest of a train of axisymmetric waves. The wave amplitude decreases with increasing distance from the source. Then we give a local analysis of axisymmetric free-surface flows with a circular ring of stagnation points. We find flows in which the surface has a discontinuity in slope with an enclosed angle of 120° all along the ring. This behaviour is consistent with the numerical solution for $F = F_c$ near the crest of the bump.

1. Introduction

Injection or withdrawal of water from a reservoir is often modelled by a source or sink below a free surface. Over the years various configurations have been considered. Some authors assume that above the source there is a stagnation point (Peregrine 1972; Mekias & Vanden-Broeck 1991; Hocking & Forbes 1992; Vanden-Broeck 1996) and others that there is a cusp (Tuck & Vanden-Broeck 1984; Hocking 1985, 1991; Vanden-Broeck & Keller 1987; Vanden-Broeck 1997; Hocking & Vanden-Broeck 1997). The solutions are usually obtained numerically, although there is an exact solution due to Sautreaux (1901) and Craya (1949) and there are some free streamline solutions when gravity is neglected (Collings 1986; Vanden-Broeck & Keller 1987; Hocking 1988). In the far field, the free surface is either flat or contains a train of waves. All these calculations are two-dimensional.

There is an interesting axisymmetric extension due to Forbes & Hocking 1990. It is a flow in fluid of infinite depth with a stagnation point above the sink (see figure 1). Hocking & Forbes used Green's theorem to formulate the problem as an integral equation. They discretized it and solved the resulting algebraic equations by Newton's method. Their results show that there is a solution for each value of the Froude number

$$F^2 = \frac{m^2}{gH^5} \tag{1.1}$$

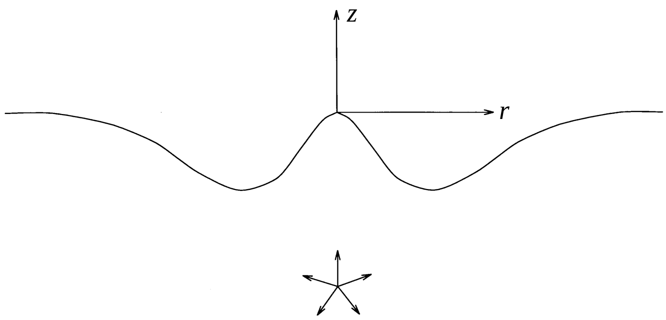

FIGURE 1. Sketch of the flow and of the coordinates.

smaller than a critical value F_c. Here m is the total flux produced by the sink, g is the acceleration due to gravity and H is the distance between the sink and the stagnation point above it. For F small, as r increases the computed surface elevation first decreases from the stagnation level, reaches a minimum, and then rises again to the stagnation level as $r \to \infty$ (see figure 1). Forbes & Hocking (1990) showed that the solution is then accurately described by an expansion in powers of F. For larger values of F, a bump appears on the surface beyond the minimum. As F increases, the height of the bump increases and it reaches the stagnation level for $F = F_c$.

The solution for $F = F_c$ is particularly interesting because it contains a circular ring of stagnation points on a free surface. In §3 we determine analytically a local solution with a circular ring of stagnation points. This solution has a discontinuity in slope at the ring, with an enclosed angle of 120°. It is an axisymmetric generalization of the classical two-dimensional solution proposed by Stokes (1847) to describe the flow near the crest of a gravity wave of maximum amplitude.

In §2, we present a numerical procedure to solve the flow problem of figure 1, similar to that of Forbes & Hocking (1990). The differences are explained in §2. We confirm their finding that a bump arises on the surface. However, by truncating the domain of computation at a sufficiently large value of r, we find that the bump is the first crest of a train of waves. As F increases, the waves grow, reaching a limiting configuration with a ring of stagnation points on the free surface when $F = F_c$. The surface near this ring of stagnation points is consistent with the 120° angle of the local analytic solution determined in §3.

2. Formulation

We consider the axisymmetric flow due to a submerged sink of strength m in water of infinite depth. We seek flows for which there is a stagnation point on the free surface just above the source (see figure 1). The fluid is assumed to be incompressible and inviscid and the flow to be irrotational. We introduce cylindrical coordinates r, θ and z with the origin at the stagnation point. The source is at $z = -H$, $r = 0$. The flow is axisymmetric about the z-axis, so that all the variables are independent of θ. We denote by $z = \zeta(r)$ the equation of the free surface.

We introduce dimensionless variables by taking H as the reference length and m/H as the reference velocity. Following Forbes & Hocking (1990), we formulate the problem in terms of the potential function ϕ. This function must satisfy Laplace's equation in the flow domain together with the conditions

$$\phi \to \frac{1}{4\pi}\frac{1}{[r^2+(z+1)^2]^{1/2}} \quad \text{as} \quad (r,z) \to (0,-1), \tag{2.1}$$

$$\phi_z = \phi_r \zeta_r \quad \text{on} \quad z = \zeta(r) \tag{2.2}$$

and

$$\tfrac{1}{2}F^2(\phi_r^2+\phi_z^2)+z = 0 \quad \text{on} \quad z=\zeta(r), \tag{2.3}$$

where F is defined in (1.1). Furthermore we require ϕ and its first derivatives to vanish at infinity. Equations (2.2) and (2.3) are the kinematic and dynamic boundary conditions. Relation (2.1) specifies the singular behaviour associated with the source.

By using Green's theorem, Forbes & Hocking (1990) derived the following integro–differential system on the free surface:

$$\tfrac{1}{2}F^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 + \zeta(s) = 0, \tag{2.4}$$

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}\zeta}{\mathrm{d}s}\right)^2 = 1, \tag{2.5}$$

$$2\pi\phi(s) = \frac{1}{[r^2(s)+(\zeta(s)+1)^2]^{1/2}} - \int_0^\infty [\phi(\sigma)-\phi(s)]k(A,B,C,\mathrm{d})\mathrm{d}\sigma\ . \tag{2.6}$$

Here

$$A = r(\sigma)\zeta'(\sigma) - r'(\sigma)[\zeta(\sigma)-\zeta(s)], \quad B = r(s)\zeta'(\sigma), \tag{2.7}$$

$$C = r^2(\sigma)+r^2(s)+[\zeta(\sigma)-\zeta(s)]^2, \quad D = 2r(s)r(\sigma), \tag{2.8}$$

$$k(A,B,C,D) = \frac{4r(\sigma)}{D(C+D)^{1/2}}\left[BK\left(\frac{2D}{C+D}\right) + \frac{AD-BC}{C-D}E\left(\frac{2D}{C+D}\right)\right], \tag{2.9}$$

and s and σ denote the arclength along the free surface In (2.9) K and E are elliptic integrals of the first and second kind respectively.

We solve the system defined by (2.4)–(2.6) numerically. The procedure is similar to the one used by Forbes & Hocking (1990). The only differences are in the finite difference formula and in the approximation of the integral in (2.6). The details can be summarized as follows. We introduce the mesh points

$$s_I = (I-1)e, \quad I = 1,\ldots,N, \tag{2.10}$$

and the corresponding unknowns

$$\phi_I = \phi(s_I), \quad I = 1,\ldots,N. \tag{2.11}$$

Next we introduce the midpoints

$$s_I^m = (s_I + s_{I+1})/2, \quad I = 1,\ldots,N-1, \tag{2.12}$$

and evaluate $\phi(s_I^m)$ and $\phi'(s_I^m)$ in terms of the unknowns by linear interpolation and centred differences. Here the prime denotes derivative with respect to s. We substitute the values of $\phi'(s_I^m)$ into (2.4) and solve for $\zeta(s_I^m)$. The values of $\zeta(s_I)$ are obtained by linear interpolation. These values are then used to calculate $\zeta'(s_I)$ by centred differences. The corresponding values $r'(s_I)$ are calculated by using (2.5). Integrating them with the trapezoidal rule gives the values of $r(s_I)$ and $r(s_I^m)$.

We satisfy (2.6) at the mesh points (2.10) $I = 2,\ldots,N-1$. The integral is

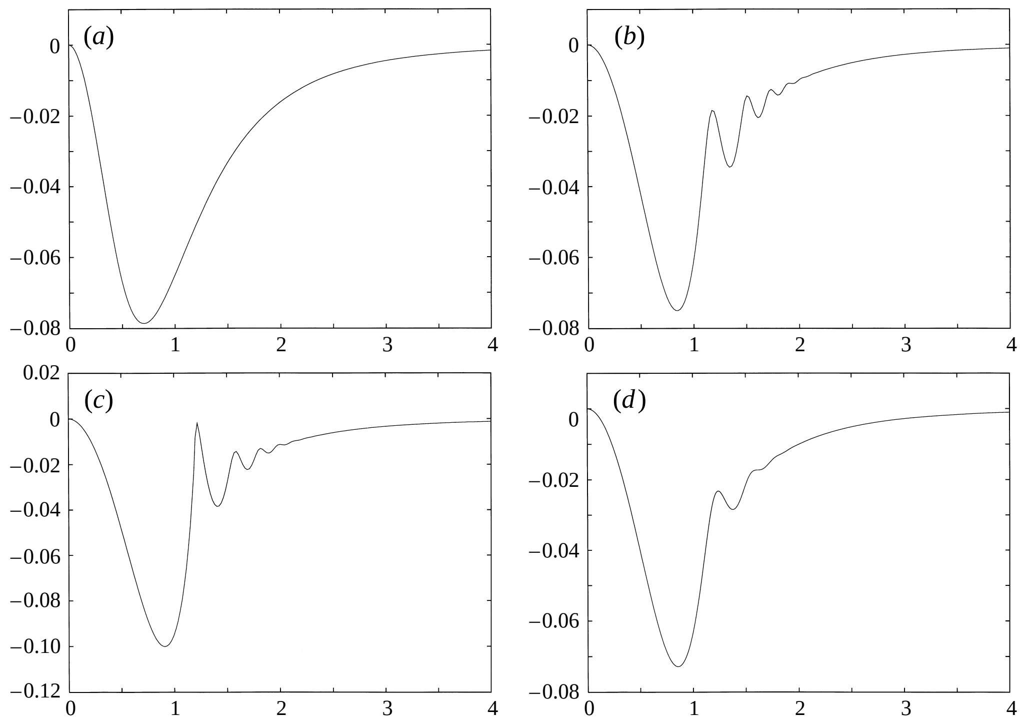

FIGURE 2. Free-surface profiles in the (r, z)-plane for (*a*) $F = 2$, (*b*) $F = 5$, (*c*) $F = F_c \approx 5.4$, (*d*) $F = 5$. The profile (*d*) differs from the one in (*b*) because the truncation value of r is not large enough.

approximated by the trapezoidal rule with a summation over the points s_I^m. This leads to $N - 2$ equations for the N unknowns (2.11). The last two equations are obtained by relating ζ_1 and ϕ_1 to ζ_I and ϕ_I, $I = 2, \ldots, 4$, by three-point interpolation formulas. This system is solved by Newton's method. Most results were obtained with $N = 600$ and $e = 0.02$. We checked that the results presented are independent of e and N by repeating the calculations with larger N and smaller e. We also repeated the calculations with other choices for the last two equations (in particular other interpolation formulas) and obtained the same results within graphical accuracy.

Typical free-surface profiles are shown in figure 2(*a–c*). Figures 2(*b*) and 2(*c*) show that in general there is a train of waves on the surface. The wave amplitude decreases as r increases. As $r \to \infty$, the free surface becomes flat and approaches the stagnation level. These results are qualitatively similar to those of Forbes & Hocking (1990). The main difference is that we find a train of waves on the free surface whereas they find one bump. The reason is that they truncated the integral equation at too small a value of s. It is necessary to truncate at a sufficiently large value $s^* = (N - 1)e$ of s for the waves to appear, and for the numerical results to be independent of s^*. This can be seen by comparing figure 2(*b*), obtained with $s^* = 12$, with figure 2(*d*), obtained with $s^* = 6$, both for $F = 5$. In figure 2(*d*), the waves are of smaller amplitude, and only the first bump is significant. The profile in figure 2(*d*) is close to that for $F = 5$ shown in figure 2 of Forbes & Hocking's paper, which was also computed with $s^* = 6$. As a check, we calculated surface profiles with $s^* > 12$ and they agreed within graphical accuracy with the one in figure 2(*b*).

We believe that there is a train of waves for each value of $F \leqslant F_c$ but for F small

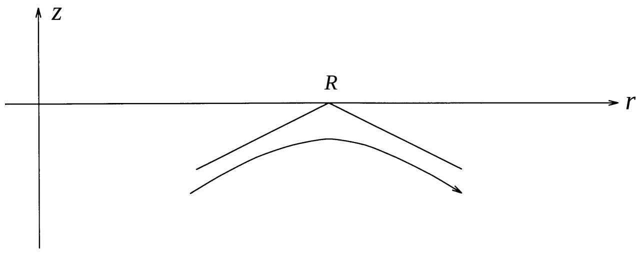

FIGURE 3. Sketch of the axisymmetric free-surface flow near a stagnation point.

they are too small to be seen on the figures. Instead, as r increases, the free-surface elevation just decreases to a minimum and then appears to rise monotonically to the stagnation level as $r \to \infty$ (see figure 2(a)). As F increases, the amplitude of the waves increases. At $F = F_c \approx 5.4$, the first crest reaches the stagnation level, and no solutions were found for $F > F_c$. (Forbes & Hocking calculated $F_c \approx 6.4$.) The first crest becomes sharper as F approaches F_c, suggesting that the slope becomes discontinuous at $F = F_c$. In the next section, we show analytically that there is indeed a local axisymmetric solution with a discontinuity in slope on the free surface. Figure 4 shows that this local solution is consistent with the numerical solution for $F = F_c$.

3. Axisymmetric solution near a ring of stagnation points on a free surface

Now we study the flow in the neighbourhood of a ring of stagnation points on a free surface (see figure 3). We assume that the flow is axisymmetric and we choose cylindrical coordinates r, z such that the ring of stagnation points is at $r = R$ and $z = 0$. As in §2, we introduce the potential function ϕ and denote the equation of the free surface by $z = \zeta(r)$. The problem is then to solve

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0 \tag{3.1}$$

with the boundary conditions

$$\phi_z = \phi_r \zeta_r \quad \text{on} \quad z = \zeta(r), \tag{3.2}$$

$$\tfrac{1}{2}(\phi_r^2 + \phi_z^2) + F^{-2} z = 0 \quad \text{on} \quad z = \zeta(r). \tag{3.3}$$

We seek a solution near the line of stagnation points $r = R, z = 0$ by writing

$$r = R + \epsilon \hat{r}, \quad z = \epsilon \hat{z}, \tag{3.4}$$

$$\zeta(r) = \epsilon \hat{\zeta}(\hat{r}) + o(\epsilon), \tag{3.5}$$

$$\phi(r, z) = \epsilon^{3/2} \hat{\phi}(\hat{r}, \hat{z}) + o(\epsilon^{3/2}), \tag{3.6}$$

where ϵ is a small positive parameter.

Substituting (3.3)–(3.6) into (3.1)–(3.3) and retaining only the leading-order terms, i.e. the terms of order $\epsilon^{1/2}$ in (3.1) and (3.2), and terms of order ϵ in (3.3), yields

$$\hat{\phi}_{\hat{r}\hat{r}} + \hat{\phi}_{\hat{z}\hat{z}} = 0, \tag{3.7}$$

$$\hat{\phi}_{\hat{z}} = \hat{\phi}_{\hat{r}} \hat{\zeta}_{\hat{r}}, \tag{3.8}$$

$$\tfrac{1}{2}(\hat{\phi}_{\hat{r}}^2 + \hat{\phi}_{\hat{z}}^2) + F^{-2} \hat{z} = 0 \quad \text{on} \quad \hat{z} = \hat{\zeta}(\hat{r})\ . \tag{3.9}$$

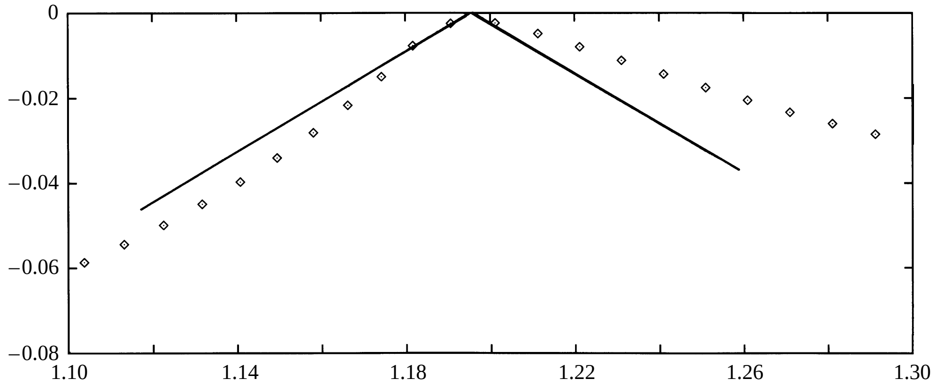

FIGURE 4. Enlargement of figure 2(c) near the first crest. The dots are the computed values ζ_I. The solid straight lines, given by (3.11), enclose an angle of 120° symmetric about the vertical.

We now seek a solution of (3.7)–(3.9) with a stagnation point on the free surface at $\hat{r} = \hat{z} = 0$. This is the classical Stokes problem in the $(\hat{r}, \hat{z})$-plane. A solution is

$$\hat{\phi} = \tfrac{2}{3}F^{-1}(\hat{r}^2 + \hat{z}^2)^{3/4} \cos\left[\tfrac{3}{2}\left(\tfrac{1}{6}\pi + \tan^{-1}(\hat{z}/\hat{r})\right)\right], \tag{3.10}$$

$$\hat{\zeta}(\hat{r}) = -\tfrac{1}{3}\sqrt{3}|\hat{r}| \,. \tag{3.11}$$

From (3.11), we see that there is a discontinuity in the slope of the free surface at $\hat{r} = 0$ with an enclosed angle of 120° symmetric about the vertical direction.

When (3.10) and (3.11) are used in (3.5) and (3.6), the parameter ϵ can be combined with $\hat{r}$ and $\hat{z}$ and the results can be written in terms of r and z. Thus ϵ was just a convenient tool for expanding ζ and ϕ in powers of distance from the stagnation point.

In figure 4 we show the free surface given by (3.11), and we also show an enlargement of the neighbourhood of the first crest in the surface profile of figure 4, calculated for $F = F_c$. Although there are not many mesh points near the crest, the numerical results are consistent with the angle of 120° symmetric about the vertical given by (3.11).

This work was supported in part by the National Science Foundation.

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